M. Math. II - Commutative Algebra
Semestral Examination : 23.11.2022
Instructor : B Sury
Maximum Marks 45; Each question carries 9 marks.
Unless specified otherwise, all rings are commutative with $1 \neq 0$.

Q 1a.
Prove that a finitely generated, projective module over a local ring must be free.

OR

Q 1b.
If $(A, \mathbf{m})$ is a local ring, and if $x_1, \cdots, x_n$ are elements of $\mathbf{m}$ whose images generate $\mathbf{m}/\mathbf{m}^2$ as a $A/\mathbf{m}$-vector space, show that $\mathbf{m}$ is generated by $x_1, \cdots, x_n$.

Q 2a.
Let A be a ring and M an A-module.
Assume the following criterion for flatness :
M is flat if the map $I \otimes_A M \to M$; $\sum a_i \otimes m_i \mapsto \sum a_i m_i$, is injective for every ideal I.
Deduce that when A is a PID, torsion-free A-modules are flat.

OR

Q 2b.
Let $\theta : M \to N$ be an A-module homomorphism. If N is finitely generated and $\theta_P : M_P \to N_P$ is onto, for a fixed prime ideal P, show that there exists $a \notin P$ such that the induced map : $M_a \to N_a$ is onto.

Q 3a.
Let A be a domain. Prove that it is integrally closed if and only if $A[X]/(f)$ is a domain for each monic irreducible polynomial $f \in A[X]$.

OR

Q 3b.
For a field K, consider the ring $A = K[X, Y]/(X^3 - Y^2)$. Prove that A is a domain which is not integrally closed.

Q 4a.

If R is a finitely generated k-algebra for a field k, prove that $Nil(R) = Jac(R)$.

OR

Q 4b.

Let $B \supset A$ be an integral extension and let $P \in Spec(A)$. If Q is an ideal of B which is maximal with respect to the property that $Q \cap A \subseteq P$, prove that Q is a prime ideal.

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Q 5a.

Let $A = \oplus_{n \geq 0} A_n$ be a graded ring generated by A_1 as an A_0-algebra. If the ring A_0 is a Noetherian and A_1 is finitely generated as an A-module, show that A is Noetherian.

OR

Q 5b.

Using Artin-Rees lemma or otherwise, prove :

If A is a Noetherian ring, I is an ideal, $a \in A$ is not a zero divisor, then there exists $n_0 > 0$ so that for all $n \geq n_0$, the relation $ab \in I^n$ implies $b \in I^{n-n_0}$.